$5^{??}$

# Problem of the Week
## Problem D and Solution
## How Many Fives?

### Problem

The product of the first seven positive integers is equal to

$$7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 5040$$

Mathematicians will write this product as 7!. This is read as "7 factorial". So, $7! = 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 5040$.

This factorial notation can be used with any positive integer. For example, $11! = 11 \times 10 \times 9 \times \cdots \times 3 \times 2 \times 1 = 39\,916\,800$. The three dots "$\cdots$" represent the product of the integers between 9 and 3.

Suppose $N = 1000!$. That is,

$$N = 1000! = 1000 \times 999 \times 998 \times 997 \times \cdots \times 3 \times 2 \times 1$$

Note that $N$ is divisible by 5, 25, 125, and 625. Each of these factors is a power of 5. That is, $5 = 5^1$, $25 = 5^2$, $125 = 5^3$, and $625 = 5^4$.

Determine the largest power of 5 that divides $N$.

## Solution

## Solution 1

In order to determine the largest power of 5 that divides $N$, we need to count the number of times the factor 5 appears in the prime factorization of $N$.

Since $N$ is equal to the product of the integers from 1 to 1000, let's first look at which of these integers are divisible by 5. The integers from 1 to 1000 that are divisible by 5 are $5, 10, 15, 20, \ldots, 990, 995, 1000$. That is, a total of $\frac{1000}{5} = 200$ integers from 1 to 1000 are divisible by 5.

Each integer that is a multiple of 25 will add an additional factor of 5, since $25 = 5 \times 5$. There are $\frac{1000}{25} = 40$ integers from 1 to 1000 that are multiples of 25. These integers give another 40 factors of 5 bringing the total to $200 + 40 = 240$.

Each integer that is a multiple of 125 will add an additional factor of 5. This is because $125 = 5 \times 5 \times 5$, and two of the factors have already been counted when we looked at 5 and 25. There are $\frac{1000}{125} = 8$ integers from 1 to 1000 that are multiples of 125. These integers give another 8 factors of 5 bringing the total to $240 + 8 = 248$.

Each integer that is a multiple of 625 will add an additional factor of 5. This is because $625 = 5 \times 5 \times 5 \times 5$, and three of the factors have already been counted when we looked at 5, 25 and 125. There is 1 integer from 1 to 1000 that is a multiple of 625, namely, 625. This integer gives another factor of 5 bringing the total to $248 + 1 = 249$.

The next power of 5 is $5^5 = 3125 > 1000$, so we have counted all factors of 5 in 1000!.

Thus, the prime factorization of $N$ contains exactly 249 factors of 5. Therefore, the largest power of 5 that divides $N$ is $5^{249}$.

## Solution 2

There are many similarities between Solution 1 and the following solution. In this solution we will divide out factors of 5 until there are none left.

1. In the integers from 1 to 1000, there are $\frac{1000}{5} = 200$ integers that are divisible by 5, namely, $5, 10, 15, \ldots, 990, 995, 1000$. If we divide each of these integers by 5, we obtain the second list $1, 2, 3, \ldots, 198, 199, 200$.
2. This second list contains $\frac{200}{5} = 40$ integers that are divisible by 5, namely, $5, 10, 15, \ldots, 190, 195, 200$. If we divide each of these integers by 5, we obtain the third list $1, 2, 3, \ldots, 38, 39, 40$.
3. This third list contains $\frac{40}{5} = 8$ integers that are divisible by 5, namely, $5, 10, 15, 20, 25, 30, 35, 40$. If we divide each of these integers by 5, we obtain the fourth list $1, 2, 3, 4, 5, 6, 7, 8$.
4. This fourth list contains 1 integer that is divisible by 5, namely the integer 5.

In total, there are $200 + 40 + 8 + 1 = 249$ factors of 5 in $1000!$. Therefore, the largest power of 5 that divides $N$ is $5^{249}$.

An interpretation of what has happened is in order. When we created the first list of multiples of 5, we discovered that there were 200 integers from 1 to 1000 that are divisible by 5. When we created the second list of multiples of 5, we were actually counting the 40 integers from 1 to 1000 that are divisible by 25. When we created the third list of multiples of 5, we were actually counting the 8 integers from 1 to 1000 that are divisible by 125. And finally, when we created the fourth list of multiples of 5, we were actually counting the 1 integer from 1 to 1000 that is divisible by 625.