



Problem of the Week

Problem E and Solution

Painting is Fund (Raising)

Problem

The POTW High School Painting Club has 30 student members, some from Grade 11 and the remainder from Grade 12. Over the year, they want to fundraise money for the school art department. Each possible pairing of students from the club will create exactly one painting together. When two Grade 11 students paint together, they will sell the painting for \$20. When a Grade 11 and a Grade 12 student paint together, they will sell the painting for \$30. When two Grade 12 students paint together, they will sell the painting for \$40.

When all the paintings are sold, the students will have raised \$13 920. How many of the 30 members of the club are Grade 11 students?

Solution

If 3 students, Student A , Student B , and Student C , are in the same grade, then there will be $(3 \times 2) \div 2 = 3$ pairings, namely AB , AC , and BC . (If we look at this using a counting argument, there would be 3 choices for the first student and for each of these choices, there would be 2 choices for the second student, a total of $3 \times 2 = 6$ pairings, namely AB , AC , BA , BC , CA , and CB . However, notice that each pairing appears twice. Since order is not important we must divide by 2, getting us 3 possible pairings.)

If 3 students, Student A , Student B , and Student C , are in Grade 11 and 2 students, Student D and Student E , are in Grade 12, then there will be $3 \times 2 = 6$ pairings of students in different grades, namely AD , AE , BD , BE , CD , and CE . (There are 3 choices for the Grade 11 student in the pairing and for each of these choices, there are 2 possibilities for the Grade 12 student. This gives a total of $3 \times 2 = 6$ pairings.)

Similar arguments will now be applied to our problem.

Let a represent the number of Grade 11 students in the club and $(30 - a)$ represent the number of Grade 12 students in the club.

Since there are a students in Grade 11 and each must paint with every other student in Grade 11, there will be $a \times (a - 1) \div 2$ paintings from pairs where both students are in Grade 11. Thus, the amount raised by paintings from pairs where both students are in Grade 11 would be

$$20 \times \left(\frac{a(a - 1)}{2} \right)$$



Similarly, since there are $(30 - a)$ students in Grade 12 and each must paint with every other student in Grade 12, there will be

$(30 - a) \times (30 - a - 1) \div 2 = (30 - a) \times (29 - a) \div 2$ paintings from pairs where both students are in Grade 12. Thus, the amount raised by paintings from pairs where both students are in Grade 12 would be

$$40 \times \left(\frac{(30 - a)(29 - a)}{2} \right)$$

Since every Grade 11 student must paint with every Grade 12 student, there will be $a \times (30 - a)$ paintings from pairs with one student from each grade. Thus, the amount raised by paintings from pairs with one student from each grade would be

$$30 \times (a(30 - a))$$

Therefore,

$$13920 = 20 \left(\frac{a(a - 1)}{2} \right) + 40 \left(\frac{(30 - a)(29 - a)}{2} \right) + 30(a(30 - a))$$

$$13920 = 10(a^2 - a) + 20(870 - 59a + a^2) + 30(30a - a^2)$$

$$1392 = (a^2 - a) + 2(870 - 59a + a^2) + 3(30a - a^2)$$

$$1392 = a^2 - a + 1740 - 118a + 2a^2 + 90a - 3a^2$$

$$1392 = -29a + 1740$$

$$29a = 348$$

$$a = 12$$

Therefore, 12 of the students in the club are in Grade 11.