$g(a) + g(b) = g(ab)$

Problem of the Week
Problem E and Solution
Sum Product Function

Problem

A function, g, has $g(2) = 5$ and $g(3) = 7$. In addition, g has the property that

$$g(a) + g(b) = g(ab)$$

for all positive integers a and b.
For example, $g(6) = g(2) + g(3) = 12$.

What is the value of $g(648)$?

Solution

We can rewrite $g(648)$ as:

$$\begin{aligned}
g(648) &= g(2 \cdot 324) \\
&= g(2) + g(324) \\
&= g(2) + g(2 \cdot 162) \\
&= g(2) + g(2) + g(162) \\
&= g(2) + g(2) + g(2 \cdot 81) \\
&= g(2) + g(2) + g(2) + g(81) \\
&= g(2) + g(2) + g(2) + g(3 \cdot 27) \\
&= g(2) + g(2) + g(2) + g(3) + g(27) \\
&= g(2) + g(2) + g(2) + g(3) + g(3 \cdot 9) \\
&= g(2) + g(2) + g(2) + g(3) + g(3) + g(9) \\
&= g(2) + g(2) + g(2) + g(3) + g(3) + g(3 \cdot 3) \\
&= g(2) + g(2) + g(2) + g(3) + g(3) + g(3) + g(3)
\end{aligned}$$

Therefore, $g(648) = 3g(2) + 4g(3) = 3(5) + 4(7) = 43$.

Note:

While this answers the question, is there actually a function that satisfies the requirements?
The answer is yes.
One function that satisfies the requirements of the problem is the function g defined by

$$g(1) = 0 \text{ and } g(2^p 3^q r) = 5p + 7q$$

for all non-negative integers p and q and all positive integers r that are not divisible by 2 or by 3. Can you see why this function satisfies the requirements?