Problem of the Week

Problem E and Solution

Dis-Card

Problem

Noah and Acacia each have nine cards, numbered from 1 to 9. Each person randomly discards one of their cards and then calculates the sum of the numbers on their remaining eight cards.

Calculate the probability that the difference between these two sums is a multiple of 4.

Solution

Let n represent the sum of the numbers on Noah's remaining eight cards, and a represent the sum of the numbers on Acacia's remaining eight cards. In order to determine the probability, we must determine the number of ways n and a can differ by a multiple of 4 and divide by the total number of ways Noah and Acacia can each discard one card.

First, we determine the total number of ways Noah and Acacia can each discard one card. Since each person has 9 cards to choose from, it follows that there are $9 \times 9 = 81$ possibilities.

The sum of the numbers on each person's original cards is $1+2+3+4+5+6+7+8+9 = \frac{9(10)}{2} = 45$. Since the cards are numbered from 1 to 9, after one card is discarded the remaining sum will be between 36 and 44. From here we present two different solutions.

Solution 1

If Noah discarded the card numbered 1, the sum of the numbers on the remaining cards would be $45 - 1 = 44$, which is n. If Acacia also discarded the card numbered 1, then $a = 44$ as well. Then $n - a = 0$, which is a multiple of 4. Acacia could also discard the cards numbered 5 or 9. Then $a = 45 - 5 = 40$ or $a = 45 - 9 = 36$, and $n - a = 4$ or $n - a = 8$, both of which are multiples of 4.

In general, if Acacia discards a card with the same number as Noah or the same number as Noah increased or decreased by some multiple of 4, then $n - a$ will be a multiple of 4. The following table summarizes the possibilities.

Number Discarded by Noah	Sum of Numbers on Noah's Remaining Cards (n)	Number Discarded by Acacia	Sum of Numbers on Acacia's Remaining Cards (a)	Value of $n - a$	Number of Values of $n - a$ that are Divisible by 4
1	44	1 or 5 or 9	44 or 40 or 36	0 or 4 or 8	3
2	43	2 or 6	43 or 39	0 or 4	2
3	42	3 or 7	42 or 38	0 or 4	2
4	41	4 or 8	41 or 37	0 or 4	2
5	40	1 or 5 or 9	44 or 40 or 36	-4 or 0 or 4	3
6	39	2 or 6	43 or 39	-4 or 0	2
7	38	3 or 7	42 or 38	-4 or 0	2
8	37	4 or 8	41 or 37	-4 or 0	2
9	36	1 or 5 or 9	44 or 40 or 36	-8 or -4 or 0	3

We find that there are $3+2+2+2+3+2+2+2+3 = 21$ different ways to discard the cards so that n and a differ by a multiple of 4.

Therefore, the probability that n and a differ by a multiple of 4 is $\frac{21}{81} = \frac{7}{27}$.

Solution 2

Let's systematically look at Noah's possible choices.

- Noah removes card 1. Then the sum of the numbers on Noah's remaining cards is $n = 45 - 1 = 44$. The sum of the numbers on Acacia's remaining cards, a, must satisfy $36 \le a \le 44$ and $44 - a$ is a multiple of 4. There are three possibilities for $a:\ 44,\ 40,\ 36$. These values of a correspond to Acacia removing cards $1,\ 5$, and 9, respectively.
- Noah removes card 2. Then the sum of the numbers on Noah's remaining cards is $n = 45 - 2 = 43$. The sum of the numbers on Acacia's remaining cards, a, must satisfy $36 \le a \le 44$ and $43 - a$ is a multiple of 4. There are two possibilities for $a:\ 43,\ 39$. These values of a correspond to Acacia removing cards 2 and 6, respectively.
- Noah removes card 3. Then the sum of the numbers on Noah's remaining cards is $n = 45 - 3 = 42$. The sum of the numbers on Acacia's remaining cards, a, must satisfy $36 \le a \le 44$ and $42 - a$ is a multiple of 4. There are two possibilities for $a:\ 42,\ 38$. These values of a correspond to Acacia removing cards 3 and 7, respectively.
- Noah removes card 4. Then the sum of the numbers on Noah's remaining cards is $n = 45 - 4 = 41$. The sum of the numbers on Acacia's remaining cards, a, must satisfy $36 \le a \le 44$ and $41 - a$ is a multiple of 4. There are two possibilities for $a:\ 41,\ 37$. These values of a correspond to Acacia removing cards 4 and 8, respectively.
- Noah removes card 5. Then the sum of the numbers on Noah's remaining cards is $n = 45 - 5 = 40$. The sum of the numbers on Acacia's remaining cards, a, must satisfy $36 \le a \le 44$ and $40 - a$ is a multiple of 4. There are three possibilities for $a:\ 44,\ 40,\ 36$. These values of a correspond to Acacia removing cards $1,\ 5$, and 9, respectively.
- Noah removes card 6. Then the sum of the numbers on Noah's remaining cards is $n = 45 - 6 = 39$. The sum of the numbers on Acacia's remaining cards, a, must satisfy $36 \le a \le 44$ and $39 - a$ is a multiple of 4. There are two possibilities for $a:\ 43,\ 39$. These values of a correspond to Acacia removing cards 2 and 6, respectively.
- Noah removes card 7. Then the sum of the numbers on Noah's remaining cards is $n = 45 - 7 = 38$. The sum of the numbers on Acacia's remaining cards, a, must satisfy $36 \le a \le 44$ and $38 - a$ is a multiple of 4. There are two possibilities for $a:\ 42,\ 38$. These values of a correspond to Acacia removing cards 3 and 7, respectively.
- Noah removes card 8. Then the sum of the numbers on Noah's remaining cards is $n = 45 - 8 = 37$. The sum of the numbers on Acacia's remaining cards, a, must satisfy $36 \le a \le 44$ and $37 - a$ is a multiple of 4. There are two possibilities for $a:\ 41,\ 37$. These values of a correspond to Acacia removing cards 4 and 8, respectively.
- Noah removes card 9. Then the sum of the numbers on Noah's remaining cards is $n = 45 - 9 = 36$. The sum of the numbers on Acacia's remaining cards, a, must satisfy $36 \le a \le 44$ and $36 - a$ is a multiple of 4. There are three possibilities for $a:\ 44,\ 40,\ 36$. These values of a correspond to Acacia removing cards $1,\ 5$, and 9, respectively.

We find that there are $3 + 2 + 2 + 2 + 3 + 2 + 2 + 2 + 3 = 21$ different ways to discard the cards so that n and a differ by a multiple of 4.

Therefore, the probability that n and a differ by a multiple of 4 is $\frac{21}{81} = \frac{7}{27}$.