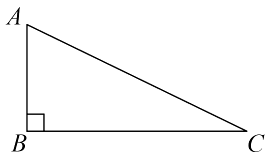



# Problem of the Week

## Problem E and Solution

## A Square in a Triangle

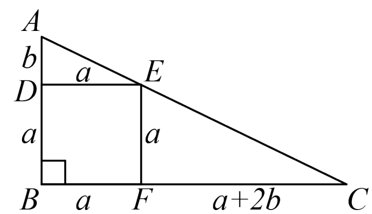
### Problem

In $\triangle ABC$, there is a right angle at $B$ and the length of $BC$ is twice the length of $AB$. In other words, $BC = 2AB$. Square $DEFB$ is drawn inside $\triangle ABC$ so that vertex $D$ is somewhere on $AB$ between $A$ and $B$, vertex $E$ is somewhere on $AC$ between $A$ and $C$, vertex $F$ is somewhere on $BC$ between $B$ and $C$, and the final vertex is at $B$.

Square $DEFB$ is called an *inscribed* square. Determine the ratio of the area of the inscribed square $DEFB$ to the area of $\triangle ABC$.

## Solution

First we draw square $DEFB$ according to the instructions in the problem. Let $DB = BF = FE = ED = a$ and $AD = b$. Since $BC = 2AB$, it follows that $BC = 2(AD+DB) = 2(a+b) = 2a+2b$. Since $BC = BF+FC$, it follows that $2a + 2b = a + FC$, so $FC = a + 2b$.




From here we present two solutions. In Solution 1, we solve the problem using similar triangles. In Solution 2, we place the diagram on the $xy$-plane and solve the problem using analytic geometry.

## Solution 1

Consider $\triangle ADE$ and $\triangle ABC$. We will first show that $\triangle ADE \sim \triangle ABC$.

Since $DEFB$ is a square, then $\angle EDB = 90^\circ$, and so $\angle EDA = 180^\circ - \angle EDB = 180^\circ - 90^\circ = 90^\circ$. Therefore, $\angle EDA = \angle ABC$. Also, $\angle DAE = \angle BAC$ since they represent the same angle. Since the angles in a triangle add to $180^\circ$, then we must also have $\angle AED = \angle ACB$.

So $\triangle ADE \sim \triangle ABC$, by Angle-Angle-Angle Triangle Similarity.

Since $\triangle ADE \sim \triangle ABC$, then corresponding side lengths are in the same ratio. In particular,

$$\begin{aligned}\frac{AD}{DE} &= \frac{AB}{BC}\\ \frac{AD}{DE} &= \frac{AB}{2AB}\\ \frac{b}{a} &= \frac{1}{2}\\ a &= 2b\end{aligned}$$

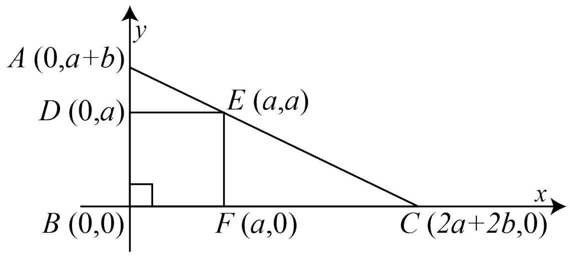
Since $BC = 2a + 2b$ and $a = 2b$, then $BC = 2(2b) + 2b = 6b$. Since $AB = a + b$ and $a = 2b$, then $AB = 2b + b = 3b$. The area of $\triangle ABC$ is $\frac{1}{2}(BC \times AB) = \frac{1}{2}(6b \times 3b) = 9b^2$.

The area of square $DEFB$ is $a \times a = a^2 = (2b)^2 = 4b^2$.

The ratio of the area of inscribed square $DEFB$ to the area of $\triangle ABC$ is $4b^2 : 9b^2 = 4 : 9$, since $b > 0$.

## Solution 2

First we place the triangle on the $xy$-plane with $B$ at $(0,0)$ and $BC$ along the positive $x$-axis. The coordinates of $D$ are $(0,a)$, the coordinates of $A$ are $(0,a+b)$, the coordinates of $F$ are $(a,0)$, the coordinates of $E$ are $(a,a)$, and the coordinates of $C$ are $(2a+2b,0)$.




Let's determine the equation of the line through $A$, $E$, and $C$.

Since this line passes through $(0,a+b)$, then we know it has $y$-intercept $a+b$.

Since it passes through $(0,a+b)$ and $(a,a)$, then the line has slope $\frac{a-(a+b)}{a-0} = -\frac{b}{a}$.

Therefore, the equation of the line through $A$, $E$, and $C$ is $y = \left(-\frac{b}{a}\right)x + a + b$.

Since $C(2a+2b,0)$ lies on this line, then substituting $x = 2a+2b$ and $y = 0$ into $y = \left(-\frac{b}{a}\right)x + a + b$ gives

$$\begin{aligned}
0 &= \left(-\frac{b}{a}\right)(2a+2b) + a + b \\
0 &= (-b)(2a+2b) + (a)(a+b) \\
0 &= -2ab - 2b^2 + a^2 + ab \\
0 &= a^2 - ab - 2b^2 \\
0 &= (a+b)(a-2b)
\end{aligned}$$

Thus, $a = -b$ or $a = 2b$. But since $a, b > 0$, then $a = -b$ is inadmissible and we must have $a = 2b$.

Since $BC = 2a+2b$ and $a = 2b$, then $BC = 2(2b)+2b = 6b$. Since $AB = a+b$ and $a = 2b$, then $AB = 2b+b = 3b$. The area of $\triangle ABC$ is $\frac{1}{2}(BC \times AB) = \frac{1}{2}(6b \times 3b) = 9b^2$.

The area of square $DEFB$ is $a \times a = a^2 = (2b)^2 = 4b^2$.

The ratio of the area of inscribed square $DEFB$ to the area of $\triangle ABC$ is $4b^2 : 9b^2 = 4 : 9$, since $b > 0$.

### Note:

From the equation $0 = (-b)(2a+2b) + (a)(a+b)$, we could have instead factored $(2a+2b)$ to obtain $0 = (-2b)(a+b) + a(a+b)$. Since $a, b > 0$, $a+b > 0$, so we could have divided out the common factor of $(a+b)$ leaving $0 = -2b + a$ which simplifies to $a = 2b$. Thus, the factoring of $a^2 - ab - 2b^2$ to determine $a = 2b$ would not have been necessary.

### Extension:

If, in the original problem, $BC = kAB$, where $k > 0$, and the square was inscribed as given, what would be the ratio of the area of square $DEFB$ to the area of $\triangle ABC$?