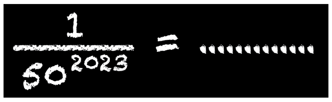

Problem of the Week

Problem E and Solution

Not Again

Problem

When $\frac{1}{50^{2023}}$ is written as a decimal, it terminates.

What is the last non-zero digit in the decimal representation of $\frac{1}{50^{2023}}$?

Solution

Our first instinct might be to use our calculator to get an idea of how the last digit behaves for the first few powers of $\frac{1}{50}$. This might work for a little while, however most calculators let us down too quickly.

Notice that $\frac{1}{50^{2023}} = \left(\frac{1}{50}\right)^{2023} = \left(\frac{1}{100} \times 2\right)^{2023} = (0.01 \times 2)^{2023} = (0.01)^{2023} \times 2^{2023}$.

The last non-zero digit in the decimal representation of $\frac{1}{50^{2023}}$ will therefore be the last non-zero digit in the decimal representation of $(0.01)^{2023}$ multiplied by the last digit of 2^{2023}.

Since the last non-zero digit in the decimal representation of $(0.01)^{2023}$ is 1, the last non-zero digit in the decimal representation of $\frac{1}{50^{2023}}$ will therefore be the last digit of 2^{2023}.

We now examine the last digit of various powers of 2:

$$\begin{array}{llll} 2^1 = \mathbf{2} & 2^2 = \mathbf{4} & 2^3 = \mathbf{8} & 2^4 = 1\mathbf{6} \\ 2^5 = 3\mathbf{2} & 2^6 = 6\mathbf{4} & 2^7 = 12\mathbf{8} & 2^8 = 25\mathbf{6} \end{array}$$

Notice that the last digit repeats every four powers of 2. It is 2, then 4, then 8, then 6. This pattern continues, and we can verify that 2^9 ends with a 2, 2^{10} ends with a 4, 2^{11} ends with an 8, 2^{12} ends with a 6, and so on. We will leave it up to the solver to explain *why* this pattern continues.

Now we need to determine the number of complete cycles of this pattern before we get to 2^{2023}. Since $\frac{2023}{4} = 505\frac{3}{4}$, it follows that there are there are 505 complete cycles. Since $505 \times 4 = 2020$, this means 2^{2020} ends with a 6, 2^{2021} ends with a 2, 2^{2022} ends with a 4, and 2^{2023} ends with an 8.

Since 2^{2023} ends with an 8, $\frac{1}{50^{2023}}$ also ends with an 8.